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Finite Element Solution for Two Dimensional Laplace Equation with Dirichlet Boundary Conditions

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Abstract

The steady state heat distribution in a plane region is modeled by two dimensional Laplace equation. In this paper Galerkin technique has been used to construct Finite Element model for two dimensional steady heat flow problem with Dirichlet boundary conditions in a rectangular domain. Results are then compared with analytic solution to check the accuracy of the developed scheme.

Key Words: Dirichlet Conditions, Finite Element Model, Galerkin Method, Laplace Equation

1. Introduction

It is conventional to solve Laplace Equation [1] in two dimension with Dirichlet conditions. In many advanced courses on electromagnetism , it is fundamental to study the solution of Laplace equation with various boundary conditions. Particularly, the Dirichlet and Neumann boundary value problems of Laplace equation are included in advanced courses [2]. Two dimensional Laplace equation with Dirichlet boundary conditions is a model equation for steady state distribution of heat in a plane region [3]. In this paper Galerkin technique has been used to develop Finite Element model for two dimensional Laplace equation with Dirichlet boundary conditions in a rectangular domain.

2. Finite Element Model

A simple case of steady state heat conduction in a rectangular domain is defined by two dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2}(x,y)+\frac{\partial^2 u}{\partial y^2}(x,y)=0 \ in \ R \qquad (1)$$

with Dirichlet conditions

$$\left.\begin{array}{l} u(x,c)=f_1(x), u(x,d)=f_2(x), a\le x\le b \\ u(a,y)=g_1(y), u(b,y)=g_2(y), c\le y\le d \end{array}\right\} \qquad (2)$$

2.1 Domain Discretization

Divide the region R into finite number of rectangular elements. Every node and every side of the rectangle must be common with adjacent elements except for sides on the boundaries. The nodes and elements are both numbered.

2.2 Interpolating Functions

Consider a rectangular element (e) with sides 'a' and 'b' as shown in figure 1 in which the nodes are numbered in the counterclockwise direction and derive the interpolation function [4] for it. Assume the interpolating polynomial in such a way that the number of terms and the number of nodes are equal in the element. Accordingly, assume

$$u^{(e)}(x,y)=c_1+c_2x+c_3y+c_4xy \qquad (3)$$

where $c_1 . i=1,2,3,4$ are constants.

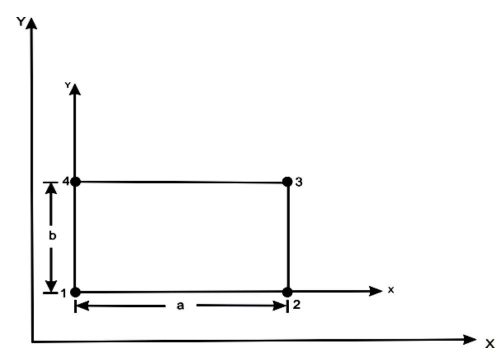

Fig.1 Rectangular element with nodes

We choose local coordinate system (x, y) to derive the interpolation functions.

The value $u(x, y)$ at each node of rectangular element is given by

$$u_1 = u^{(e)}(0,0) = c_1 \quad (4a)$$
$$u_2 = u^{(e)}(a,0) = c_1 + c_2 a$$
$$u_2 = u^{(e)}(a,0) = c_1 + c_2 a \quad (4b)$$
$$u_3 = u^{(e)}(a,b) = c_1 + c_2 a + c_3 b + c_4 ab \quad (4c)$$
$$u_4 = u^{(e)}(a,b) = c_1 + c_3 b \quad (4d)$$

Solving equations (4) for $c_i . i = 1, 2, 3, 4$, we obtain

$$c_1 = u_1$$

$$c_2 = \frac{u_2 - u_1}{a}$$

$$c_3 = \frac{u_4 - u_1}{b}$$

$$c_4 = \frac{u_3 - u_4 + u_1 - u_2}{ab}$$

Substituting the values of $c_i . i = 1, 2, 3, 4$ in equation (3)

$$u^{(e)}(x, y) = u_1 + \left(\frac{u_2 - u_1}{a}\right)x + \left(\frac{u_4 - u_1}{b}\right)y + \left(\frac{u_3 - u_4 + u_1 - u_2}{ab}\right)xy$$

Collecting the coefficients of u_1, u_2, u_3 and u_4 in the above equation, we have

$$u^{(e)}(x, y) = \left(1 - \frac{x}{a}\right)\left(1 - \frac{y}{b}\right)u_1 + \frac{x}{a}\left(1 - \frac{y}{b}\right)u_2 + \left(\frac{xy}{ab}\right)u_3 + \left(1 - \frac{x}{a}\right)\frac{y}{b}\, u_4$$

$$u^{(e)}(x, y) = \sum_{i=1}^{4} N_i^{(e)}\, u_i^{(e)}$$

where

$$N_1^{(e)} = \left(1 - \frac{x}{a}\right)\left(1 - \frac{y}{b}\right)$$

$$N_2^{(e)} = \frac{x}{a}\left(1 - \frac{y}{b}\right)$$

$$N_3^{(e)} = \frac{xy}{ab}$$

$$N_4^{(e)} = \left(1 - \frac{x}{a}\right)\frac{y}{b}$$

2.3 Element Equations

The Galerkin [5] approach is applied to construct Finite Element model of the equation (1) over the element (e). Substituting $u^{(e)}(x, y)$ into the equation (1) gives the residual

$$E^{(e)}(x, y) = \frac{\partial^2 u^{(e)}}{\partial x^2} + \frac{\partial^2 u^{(e)}}{\partial y^2}$$

Then equating the weighted residual integral to zero gives

$$\iint_R W\left(\frac{\partial^2 u^{(e)}}{\partial x^2} + \frac{\partial^2 u^{(e)}}{\partial y^2}\right)dxdy = 0 \quad (6)$$

where $W \; x, y$ is the general weighting function.

$$\iint_R W\frac{\partial^2 u^{(e)}}{\partial x^2}dxdy + \iint_R W\frac{\partial^2 u^{(e)}}{\partial y^2}dxdy = 0 \quad (7)$$

Since

$$\frac{\partial}{\partial x}\left(W\frac{\partial u^{(e)}}{\partial x}\right) = \frac{\partial W}{\partial x}\frac{\partial u^{(e)}}{\partial x} + W\frac{\partial^2 u^{(e)}}{\partial x^2}$$

Therefore

$$W\frac{\partial^2 u^{(e)}}{\partial x^2} = \frac{\partial}{\partial x}\left(W\frac{\partial u^{(e)}}{\partial x}\right) - \frac{\partial W}{\partial x}\frac{\partial u^{(e)}}{\partial x}$$

Similarly

$$W\frac{\partial^2 u^{(e)}}{\partial y^2} = \frac{\partial}{\partial y}\left(W\frac{\partial u^{(e)}}{\partial y}\right) - \frac{\partial W}{\partial y}\frac{\partial u^{(e)}}{\partial y}$$

Substituting in equation (7)

$$\iint_R \frac{\partial}{\partial y}\left(W\frac{\partial u^{(e)}}{\partial y}\right)dxdy = \oint_{\partial R} W + \frac{\partial u^{(e)}}{\partial y} n_y ds$$

$$\iint_R \frac{\partial}{\partial y}\left(W\frac{\partial u^{(e)}}{\partial y}\right)dxdy = \oint_{\partial R} W \frac{\partial u^{(e)}}{\partial y} n_y ds$$

$$\iint_{R^{(e)}} \frac{\partial}{\partial x}\left(W\frac{\partial u^{(e)}}{\partial x}\right)dxdy = \oint_{\partial R^{(e)}} W \frac{\partial u^{(e)}}{\partial x} n_x ds$$

Using gradient theorem [4]to 1st and 2nd integrals in equation (8) to transform into line integrals

$$\iint_{R^{(e)}} \left(\frac{\partial W}{\partial x}\frac{\partial u^{(e)}}{\partial x} + \frac{\partial W}{\partial y}\frac{\partial u^{(e)}}{\partial y}\right)dxdy = 0 \tag{8}$$

where $\partial R^{(e)}$ is the border of the element (e), s is the curvilinear coordinate on the boundary and $n_y = \cos(\hat{n}, \hat{j})$ and $n_y = \cos(\hat{n}, \hat{j})$ are direction cosines of the outward unit normal of the boundary.

Substituting in equation (8)

$$\oint_{\partial R} W \frac{\partial u^{(e)}}{\partial x} n_y ds + \oint_{\partial R} W \frac{\partial u^{(e)}}{\partial y} n_y ds$$

$$\iint_{R^{(e)}} \left(\frac{\partial W}{\partial x}\frac{\partial u^{(e)}}{\partial x} + \frac{\partial W}{\partial y}\frac{\partial u^{(e)}}{\partial y}\right)dxdy = 0$$

$$\oint_{\partial R} W \frac{\partial u^{(e)}}{\partial n} ds$$

$$\iint_{R^{(e)}} \left(\frac{\partial W}{\partial x}\frac{\partial u^{(e)}}{\partial x} + \frac{\partial W}{\partial y}\frac{\partial u^{(e)}}{\partial y}\right)dxdy - \iint_{R^{(e)}} W f dxdy = 0 \tag{9}$$

where

$$\frac{\partial u^{(e)}}{\partial n} = \frac{\partial u^{(e)}}{\partial x} n_x + \frac{\partial u^{(e)}}{\partial y} n_y$$

and n is the unit outward normal.

For Dirichlet boundary conditions, $u(x, y)$ is specified on the boundary, and the line integral

$$\oint_{\partial R^{(e)}} W \frac{\partial u^{(e)}}{\partial n} ds$$

is neglected. So equation (9) becomes

$$\iint_{R^{(e)}} \left(\frac{\partial W}{\partial x}\frac{\partial u^{(e)}}{\partial x} + \frac{\partial W}{\partial y}\frac{\partial u^{(e)}}{\partial y}\right) = 0 \tag{10}$$

The evaluation of equation (10) requires the function $u^{(e)}(x, y)$ and its both partial derivatives. Differentiating equation (5) gives

$$\frac{\partial u^{(e)}}{\partial x} = \left(\frac{y}{ab} - \frac{1}{a}\right)u_1 + \left(\frac{1}{a} - \frac{y}{ab}\right)u_2 + \left(\frac{y}{ab}\right)u_3 - \left(\frac{y}{ab}\right)u_4 \tag{11a}$$

$$\frac{\partial u^{(e)}}{\partial y} = \left(\frac{x}{ab} - \frac{1}{b}\right)u_1 + \left(\frac{x}{ab}\right)u_2 + \left(\frac{x}{ab}\right)u_3 + \left(\frac{1}{b} - \frac{y}{ab}\right)u_4 \tag{11b}$$

Substituting equation (11) into equation (10)

$$\iint_{R^{(e)}} \frac{\partial W}{\partial x}\left[\left(\frac{y}{ab} + \frac{1}{a}\right)u_1 + \left(\frac{1}{a} - \frac{y}{ab}\right)u_2 + \frac{y}{ab}u_3 - \frac{y}{ab}u_4\right]dxdy$$

$$\iint_{R^{(e)}} \frac{\partial W}{\partial x}\left[\left(\frac{y}{ab} - \frac{1}{a}\right)u_1 + \left(\frac{1}{a} - \frac{y}{ab}\right)u_2 + \frac{y}{ab}u_3 - \frac{y}{ab}u_4\right]dxdy$$

$$\iint_{R^{(e)}} \frac{\partial W}{\partial x}\left[\left(\frac{x}{ab} - \frac{1}{b}\right)u_1 - \frac{x}{ab}u_2 + \frac{x}{ab}u_3 + \left(\frac{1}{b} - \frac{x}{ab}\right)u_4\right]dxdy \tag{12}$$

In Galerkin weighted residual approach, the weighting factors are chosen to be shape functions i.e.

$$W_i = N_i^{(e)},\ i = 1, 2, 3, 4$$

Let's evaluate equation (12) for

$$W_i(x,y) = N_i^{(e)}(x,y) = \left(1-\frac{x}{a}\right)\left(1-\frac{y}{b}\right) \qquad (13)$$

Differentiating equation (13) with respect to x and y

$$\frac{\partial w_1}{\partial x} = \left(\frac{y}{ab}-\frac{1}{a}\right) \qquad (14a)$$

$$\frac{\partial w_1}{\partial y} = \left(\frac{x}{ab}-\frac{1}{b}\right) \qquad (14b)$$

Substituting equation (14) into equation (12)

$$\int_0^b\int_0^a\left(\frac{y}{ab}-\frac{1}{a}\right)\left[\left(\frac{y}{ab}-\frac{1}{a}\right)u_1+\left(\frac{1}{a}-\frac{y}{ab}\right)u_2 + \frac{y}{ab}u_3-\frac{y}{ab}u_4\right]dxdy$$

$$+\int_0^b\int_0^a\left(\frac{x}{ab}-\frac{1}{b}\right)\left[\left(\frac{x}{ab}-\frac{1}{b}\right)u_1-\frac{x}{ab}u_2+ \frac{x}{ab}u_3+\left(\frac{1}{b}-\frac{x}{ab}\right)u_4\right]dxdy$$

$$+\int_0^b\int_0^a\left(\frac{y}{ab}-\frac{1}{a}\right)\left[\left(\frac{y}{ab}-\frac{1}{a}\right)u_1+ \left(\frac{1}{a}-\frac{y}{ab}\right)u_2+\frac{y}{ab}u_3-\frac{y}{ab}u_4\right]dxdy$$

$$\int_0^b\int_0^a\left(\frac{x}{ab}-\frac{1}{b}\right) \left[\left(\frac{x}{ab}-\frac{1}{b}\right)u_1-\frac{x}{ab}u_2+\frac{x}{ab}u_3+\left(\frac{1}{b}-\frac{y}{ab}\right)u_4\right]dxdy$$

$$\Rightarrow \frac{b}{3a}u_1-\frac{b}{3a}u_2-\frac{b}{6a}u_3+\frac{b}{3a}u_4 + \frac{a}{3b}u_1+\frac{a}{6b}u_2-\frac{a}{6b}u_3-\frac{a}{3b}u_4 = 0$$

$$\Rightarrow \left(\frac{b}{3a}+\frac{a}{3b}\right)u_1+\left(\frac{a}{6b}+\frac{b}{3a}\right)u_2 + \left(-\frac{b}{6a}+\frac{a}{6b}\right)u_3+\left(\frac{b}{6a}-\frac{a}{3b}\right)u_4 = 0$$

$$\Rightarrow \frac{1}{6ab}2(a^2+b^2)u_1+\frac{1}{6ab}(a^2-2b^2)u_2 - \frac{1}{6ab}(a^2+b^2)u_3+\frac{1}{6ab}(b^2-2a^2)u_4 = 0 \quad (16a)$$

Similarly for

$W_2 = N_2$, $W_3 = N_3$ and $W_4 = N_4$

$$\frac{1}{6ab}(a^2-2b^2)u_1+\frac{1}{6ab}2(a^2+b^2)u_2 + \frac{1}{6ab}(b^2-2a^2)u_3-\frac{1}{6ab}(a^2+b^2)u_4 = 0 \quad (16b)$$

$$\frac{1}{6ab}(a^2+b^2)u_1+\frac{1}{6ab}(b^2-2a^2)u_2 + \frac{1}{6ab}2(a^2+b^2)u_3+\frac{1}{6ab}(a^2-2b^2)u_4 = 0 \quad (16c)$$

$$\frac{1}{6ab}(b^2-2a^2)u_1-\frac{1}{6ab}(a^2+b^2)u_2 + \frac{1}{6ab}(a^2-2b^2)u_3+\frac{1}{6ab}2(a^2+b^2)u_4 = 0 \quad (16d)$$

The above equations (16a) to (16d) can be written in matrix form

$$\left[K^{(e)}\right]\left\{u^{(e)}\right\} = \left\{F^{(e)}\right\}$$

where

$$\left[K^{(e)}\right] = \frac{1}{6ab}\begin{bmatrix} 2(a^2+b^2) & a^2-2b^2 & -(a^2+b^2) & b^2-2a^2 \\ a^2-2b^2 & 2(a^2+b^2) & b^2-2a^2 & -(a^2+b^2) \\ (a^2+b^2) & b^2-2a^2 & 2(a^2+b^2) & a^2-2b^2 \\ b^2-2a^2 & -(a^2+b^2) & a^2-2b^2 & 2(a^2+b^2) \end{bmatrix}$$

$$\{u^{(e)}\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \quad \text{and} \quad \{F^{(e)}\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Observe that $[K^{(e)}]$ is symmetrical i.e.

$$K_{ij}^{(e)} = K_{ji}^{(e)}$$

2.3.1 *Assembly of Element Equations*

There are four equations for every element. Since some or all nodes of element (e) are shared with other elements, the u-value for a shared node appears in the equations of all elements that shared the nodes. Combining all the element equations we will get a global system coefficient matrix. This matrix has rows and columns as there are number of nodes. We will assemble the system matrix in the following way.

Suppose there are 'n' nodes in the system. Label the nodes in order from 1 to n. Associate the number of each node with row and column of every element matrix where the u-value for that node appears on the diagonal. In system matrix [6], the node numbers are assigned to rows and columns in a manner like one described above.

We get the entry in row i and column j of the system matrix by adding the values from row i of every element matrix that has row i, then adding these in the columns where the column node number match. After the assembly of local systems, the global system of equations is of the form

$$[K]\{U\} = \{F\}$$

2.4 Adjusting for Dirichlet Conditions

The u-values are specified for all the nodes on the boundary. We substitute the known values in every equation where it appears and shift them on right hand side of the corresponding equation i.e. for a particular node n, all the values in column 'n' of the matrix are multiplied by the known value and subtract the result from right hand side of the corresponding row. The equations corresponding to the known nodes are removed from the system. Now our system has only equations involving unknown nodal values. This completes the adjustment of boundary conditions for the system equations and now the system is ready to solve.

2.5 Solution of Global System

Solve the system of equations for unknown u-values using an iterative method. These values are approximate solutions at the nodes. If the approximations to $u(x, y)$ at intermediate points in the region are needed, we obtain them by using linear interpolating relations.

3. Test Problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad 0 < x < 0.5 \quad 0 < y < 0.5$$

$$u(x,0) = 0, u(x,0.5) = 200\,x \qquad 0 \le x \le 0.5$$

$$u(0,y) = 0, u(0.5,y) = 200\,y \qquad 0 \le y \le 0.5$$

Exact solution is $u(x, y) = 400\,xy$

4. Conclusion

In Figure 2 surface indicates the exact solution of Laplace equation while dots show the numerical solution obtained using FEM. It is clear from the plot that results obtained by FEM are very close to exact solution.

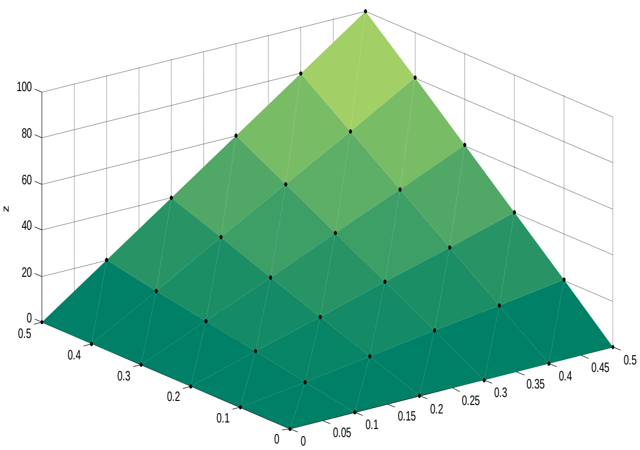

Fig. 2

X	Y	FEM Solution	Exact Solution	Absolute Error
0.0000	0.0000	0.000000	0.000000	0.000000
0.1000	0.0000	0.000000	0.000000	0.000000
0.2000	0.0000	0.000000	0.000000	0.000000
0.3000	0.0000	0.000000	0.000000	0.000000
0.4000	0.0000	0.000000	0.000000	0.000000
0.5000	0.0000	0.000000	0.000000	0.000000
0.0000	0.1000	0.000000	0.000000	0.000000
0.1000	0.1000	4.000000	4.000000	0.000000
0.2000	0.1000	8.000000	8.000000	0.000000
0.3000	0.1000	12.00000	12.00000	0.000000
0.4000	0.1000	16.00000	16.00000	0.000000
0.5000	0.1000	20.00000	20.00000	0.000000
0.0000	0.2000	0.000000	0.000000	0.000000
0.1000	0.2000	8.000000	8.000000	0.000000
0.2000	0.2000	16.00000	16.00000	0.000000
0.3000	0.2000	24.00000	24.00000	0.000000
0.4000	0.2000	32.00000	32.00000	0.000000
0.5000	0.2000	40.00000	40.00000	0.000000
0.0000	0.3000	0.000000	0.000000	0.000000
0.1000	0.3000	12.00000	12.00000	0.000000
0.2000	0.3000	24.00000	24.00000	0.000000
0.3000	0.3000	36.00000	36.00000	0.000000
0.4000	0.3000	48.00000	48.00000	0.000000
0.5000	0.3000	60.00000	60.00000	0.000000
0.0000	0.4000	0.000000	0.000000	0.000000
0.1000	0.4000	16.00000	16.00000	0.000000
0.2000	0.4000	32.00000	32.00000	0.000000
0.3000	0.4000	48.00000	48.00000	0.000000
0.4000	0.4000	64.00000	64.00000	0.000000
0.5000	0.4000	80.00000	80.00000	0.000000
0.0000	0.5000	0.000000	0.000000	0.000000
0.1000	0.5000	20.00000	20.00000	0.000000
0.2000	0.5000	40,00000	40,00000	0.000000
0.3000	0.5000	60.00000	60.00000	0.000000
0.4000	0.5000	0.800000	0.800000	0.000000
0.5000	0.5000	100.0000	100.0000	0.000000

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