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New Inequalities of Hadamard-type via *h*-Convexity

Muhammad Iqbal[1*], M. Iqbal Bhatti[1] and S. Hussain[1]

1. Department of Mathmnematics, University of Engineering & Technology, Lahore.
* Corresponding Author: *miqbal.bki@gmail.com*

Abstract

Some new Hermite-Hadamard's inequalities for h-*convex functions are proved, generalizing some results in* [1, 3, 6] *and unifying a number of known results. Some new applications for special means of real numbers are also deduced.*

Key Words: Hermite-Hadamard inequality, Hölder's inequality, Convex and *h*-convex functions.

1. Introduction and Preliminaries

If a function $f:[a,b]\to\Re$ is convex, then

$$f\left(\tfrac{a+b}{2}\right)\le \tfrac{1}{b-a}\int_a^b f(x)dx \le \tfrac{f(a)+f(b)}{2} \qquad (1)$$

is known as the Hermite-Hadamard inequality. For concave function *f*, the above order is reversed. Inequality (1) is refined, extended, generalized and new proofs are given in [1, 2, 3, 5, 7, 8].

Now we present definitions, theorems and results that we apply in this paper.

Definition 1. [1] *Let I* be an interval of real numbers. A function $f:l$ is said to be convex if for all $x, y\in l$ and $t\in$ [0, 1]

$$f\{tx+(1-t)y\}\le tf(x)+(1-t)f(y)$$

f is said to be concave, if the above inequality is reversed.

Definition 2. [4] *A non-negative function* $f:l\subseteq\mathbb{R}\to\Re$ is said to Godunova-Levin function (or *f* is said to belong to class *Q*(*l*))

if, f or all $x, y\in l$ and $t\in(0, 1)$

$$f\left(tx+(1-t)y\right)\le \frac{f(x)}{t}+\frac{f(y)}{1-t}$$

It may be noted that this class contained all non-negative monotone and non-negative convex functions .

Definition 3. [2] A function $f:[0,\infty)\to[0,\infty)$ is a function of P type (or that *f* belongs to the class P(l)) if, for all $x, y\in[0,\infty)$ and $t\in[0, 1]$

$$f\left(tx+(1-t)y\right)\le f(x)+f(y)$$

Definition 4: [1, p.288] A function $f:[0,\infty)\to\Re$ is said to be s-convex function in the second sense (or $f\in K_S^2$) if for all $x, y\in[0,\infty)$, $t\in[0, 1]$ and $s\in[0,1]$, the following inequality holds:

$$f\left(tx+(1-t)y\right)\le t^s f(x)+(1-t)^s f)y)$$

Obviously, 1-convex function is convex.

Definition 5 [10] Let *I, J* be intervals in $\Re, (0,1)\subseteq J$ and let $h:J\subseteq\mathbb{R}\to\Re$ be a non-negative function, $h\not\equiv 0$. A non-negative function $f:l\to\Re$ is called *h*-convex function (or *f* belongs to the class *SX*, (*h*, *l*)), if for all $x, y\in l$ and $t\in(0, 1)$, the inequality

$$f\left(tx+(1-t)y\right)\le h(t)f(x)+h(1-t)f(y)$$

holds

If the inequality is reversed then *f* is said to be *h*-concave and in this case *f* belongs to the class SV(h,l).

Remark 1. If *h*(*t*) = *t*, then all the non-negative convex functions belong to the class SX(h, l) and all non-negative concave functions belong to the class SV (h, l).

If $h(t)=\frac{1}{t}$, then *SX*(*h; l*) = *Q*(*l*).

If $h(t)=1$, then *SX*(*h; l*) $\supseteq$ P(*l*).

If $h(t)=t^s$, where $s\in(0, 1)$, then $SX(h, l)\supseteq K_s^2$

In [8] some new Hadamard-type inequalities for h-convex functions are discussed by authors.

In [9] Sarikaya et. al. established a Hermite-Hadamard inequality for h--convex functions in as:

Theorem 1. Let $f\in SX(h, l)$, $a, b \in l$ with $a < b$ and $f\in L[a; b]$. Then

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \qquad (2)$$

$$[f(a)+f(b)]\int_0^1 h(t)dt$$

In this article, we obtain new inequalities of Hermite-Hadamard's type for functions belong to class $SX(h, l)$. Finally, we have given some applications for special Means of real numbers.

2. Main Results

Lemma 1. Let $f: l \subseteq \mathbb{R} \to \Re$ be a differentiable function on $l^o; a,b\in l^o$ with $a<b$. If $f' \in L[a,b]$ and for all λ, $\mu > 0$, then

$$\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{\mu a+\lambda b}{\lambda+\mu}\right) = \frac{b-a}{(\lambda+\mu)^2}\times$$

$$\left[\lambda^2\int_0^1 (t-1) f'\left(ta+(1-t)\frac{\mu a+\lambda b}{\lambda+\mu}\right)dt + \right.$$

$$\left.\mu^2\int_0^1 (1-t) f'\left(tb+(1-t)\frac{\mu a+\lambda b}{\lambda+\mu}\right)dt\right] \qquad (3)$$

Proof

Let

$$l_1 = \frac{\lambda^2(b-a)}{(\lambda+\mu)^2}\int_0^1 (t-1) f'(ta+$$

$$\left(\frac{(1-t)(\mu a+\lambda b)}{\lambda+\mu}\right)dt$$

Integrating by parts and making suitable substitutions

$$l_1 = -\frac{\lambda}{\lambda+\mu} f\left(\tfrac{\mu a+\lambda b}{\lambda+\mu}\right) + \tfrac{\lambda}{\lambda+\mu}\int_0^1 f\left(ta+(1-t)\tfrac{\mu a+\lambda b}{\lambda+\mu}\right)dt$$

$$= -\frac{\lambda}{\lambda+\mu} f\left(\frac{\mu a+\lambda b}{\lambda+\mu}\right) + \frac{1}{b-a}\int_a^{\frac{\mu a+\lambda b}{\lambda+\mu}} f(x)dx \qquad (4)$$

Similarly

$$l_2 = \frac{\mu^2(b-a)}{(\lambda+\mu)^2}\int_0^1 (1-t) f'(tb+$$

$$(1-t)\frac{\mu a+\lambda b}{\lambda+\mu}\Bigg)dt$$

$$= -\frac{\mu}{\lambda+\mu} f\left(\frac{\mu a+\lambda b}{\lambda+\mu}\right) + \frac{1}{b-a}\int_{\frac{\mu a+\lambda b}{\lambda+\mu}}^b f(x)dx \qquad (5)$$

Adding (4) and (5) we obtained (3)

Lemma 2. Let the conditions of Lemma 1 be satisfied, then

$$\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu} - \frac{1}{b-a}\int_a^b f(x)dx =)b-a)$$

$$\int_0^1\left[\frac{\mu}{\lambda+\mu} - t\right] f'(ta+(1-t)b)\,dt \qquad (6)$$

Proof.

Integrating by parts and making suitable substitutions on *RHS* of (6), we get *LHS.*

Remark 2. Setting $\lambda = \mu \neq 0$, Lemma 2 coincides with [3, Lemma 2.1].

Theorem 2. Let $f: l \subseteq \mathbb{R} \to \Re$ be a differentiable function on l^o, such that $f' \in L[a,b]$ for a, b $\in$ l^o with $a < b$. If $|f'| \in SX(h,l)$ and $\lambda, \mu > 0$, then

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{\mu a+\lambda b}{\lambda+\mu}\right)\right| \le \frac{b-a}{(\lambda+\mu)^2}\times$$

$$[\{\lambda^2 |f'(a)| + \mu^2 |f'(b)|\}\int_0^1 (1-t)h(t)dt +$$

$$(\lambda^2+\mu^2)\left|f'\left(\frac{\mu a+\lambda b}{\lambda+\mu}\right)\right|\left|\int_0^1 t\,h(t)\,dt\right] \qquad (7)$$

Proof.

By taking modulus on both sides of (3)

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{\mu a+\lambda b}{\lambda+\mu}\right)\right| \le \frac{b-a}{(\lambda+\mu)^2}\times$$

$$\left[\lambda^2\int_0^1 (1-t)\left|f'\left(ta+(1-t)\frac{\mu a+\lambda b}{\lambda+\mu}\right)\right|dt + \right.$$

$$\mu^2 \int_0^1 (1-t)\left| f'\left(tb+(1-t)\frac{\mu a+\lambda b}{\lambda+\mu}\right)\right| dt \Bigg] \quad (8)$$

By *h*-convexity of $|f'|$

$$\int_0^1 (1-t)\left| f'\left(ta+(1-t)\frac{\mu a+\lambda b}{\lambda+\mu}\right)\right| dt \le$$

$$|f'(a)| \int_0^1 (1-t)h(t)dt + \left| f'\left(\frac{\mu a+\lambda b}{\lambda+\mu}\right)\right|$$

$$\times \int_0^1 (1-t)h(1-t)dt$$

and

$$\int_0^1 (1-t)\left| f'\left(tb+(1-t)\frac{\mu a+\lambda b}{\lambda+\mu}\right)\right| dt \le$$

$$|f'(b)| \int_0^1 (1-t)h(t)dt + \left| f'\left(\frac{\mu a+\lambda b}{\lambda+\mu}\right)\right|$$

$$\times \int_0^1 (1-t)h(1-t)dt$$

From here (7) follows.

Corollary 1. Under the assumptions of Theorem 2 for h(t) = t and $\lambda = \mu \neq 0$

$$\left| \frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{b-a}{24} \times$$

$$\left[|f'(a)| + 4\left| f'\left(\frac{a+b}{2}\right)\right| + f'(b) \right]$$

Here by applying convexity of $|f'|$ on the middle factor we obtained [6, Theorem 2.2].

Theorem 3. Let $f: I \subseteq \mathbb{R}$ be a differentiable function on I^o, such that $f'' \in L[a,b]$, for $a,b \in I^o$ with $a<b$ If $|f'|^q \in SX(h,I)$ with $p = \frac{q}{q-1}$ for $q>1$ and $\lambda, \mu > 0$, then

$$\left| \frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{\mu a+\lambda b}{\lambda+\mu}\right)\right| \le \frac{b-a}{\sqrt[p]{2}\,(\lambda+\mu)^2} \times$$

$$\Bigg[\lambda^2 \{ |f'(a)|^q \int_0^1 (1-t)h(t)dt + \left| f'\left(\frac{\mu a+\lambda b}{\lambda+\mu}\right)\right|^q$$

$$\int_0^1 th(t)dt\}^{1/q} + \mu^2 \{ |f'(b)|^q \int_0^1 (1-t)h(t)dt$$

$$+ \left| f'\left(\frac{\mu a+\lambda b}{\lambda+\mu}\right)\right|^q \int_0^1 th(t)\,dt\}^{1/q} \Bigg] \quad (9)$$

Proof.

By Hölder's inequality and h-convexity of $|f'|^q$:

$$\int_0^1 (1-t)\left| f'\left(ta+(1-t)\frac{\mu a+\lambda b}{\lambda+\mu}\right)\right| dt \le \frac{1}{\sqrt[p]{2}}$$

$$\times \left\{ |f'(a)| \int_0^1 (1-t)h(t)dt + \left| f'\left(\frac{\mu a+\lambda b}{\lambda+\mu}\right)\right|^q \right.$$

$$\left. \times \int_0^1 th(t)dt \right\}^{1/q} \quad (10)$$

Analogously

$$\int_0^1 (1-t)\left| f'\left(tb+(1-t)\frac{\mu a+\lambda b}{\lambda+\mu}\right)\right| dt \le \frac{1}{\sqrt[p]{2}}$$

$$\times \left\{ |f'(b)| \int_0^1 (1-t)h(t)dt + \left| f'\left(\frac{\mu a+\lambda b}{\lambda+\mu}\right)\right|^q \right.$$

$$\left. \times \int_0^1 th(t)dt \right\}^{1/q} \quad (11)$$

A combination of (8), (10) and (11) yields (9).

Corollary 2. Under the assumptions of Theorem 3 for *h*(*t*) = *t* and $\lambda = \mu \neq 0$

$$\left| \frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{b-a}{8\times\sqrt[q]{3}} \times$$

$$\left[\sqrt[q]{|f'(a)|^q + 2\left| f'\left(\frac{a+b}{2}\right)\right|^q} + \right.$$

$$\left. \sqrt[q]{|f'(b)|^q + 2\left| f'\left(\frac{a+b}{2}\right)\right|^q} \right]$$

Theorem 4. Let the assumptions of Theorem 3 be satisfied, then

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{\mu a+\lambda b}{\lambda+\mu}\right)\right| \le \frac{b-a}{\sqrt[p]{p+1}(\lambda+\mu)^2}\times$$

$$\left[\lambda^2\left\{|f'(a)|^q + \left|f'\left(\frac{\mu a+\lambda b}{\lambda+\mu}\right)\right|^q\right\}^{1/q} + \mu^2\times\right.$$

$$\left.\left\{|f'(b)|^q + \left|f'\left(\frac{\mu a+\lambda b}{\lambda+\mu}\right)\right|^q\right\}^{1/q}\right]\left(\int_0^1 h(t)dt\right)^{1/q} \qquad (12)$$

Proof

By Hölder's inequality and *h*-convexity of $|f'|^q$:

$$\int_0^1 (1-t)\left|f'\left(ta+(1-t)\frac{\mu a+\lambda b}{\lambda+\mu}\right)\right|dt \le$$

$$\frac{1}{\sqrt[p]{p+1}}\sqrt[q]{\left(|f'(a)|^q + \left|f'\left(\frac{\mu a+\lambda b}{\lambda+\mu}\right)\right|^q \int_0^1 h(t)dt\right)} \quad (13)$$

Analogously

$$\int_0^1 (1-t)\left|f'\left(tb+(1-t)\frac{\mu a+\lambda b}{\lambda+\mu}\right)\right|dt \le$$

$$\frac{1}{\sqrt[p]{p+1}}\sqrt[q]{\left(|f'(b)|^q + \left|f'\left(\frac{\mu a+\lambda b}{\lambda+\mu}\right)\right|^q \int_0^1 h(t)dt\right)} \quad (14)$$

A combination of (8), (13) and (14) yields (12).

Corollary 3. Under the assumptions of Theorem 3 for h(t) = t and λ = μ ≠ 0.

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{b-a}{8}\times\sqrt[p]{\frac{2}{p+1}}$$

$$\times\left[\sqrt[q]{|f'(a)|^q + 2\left|f'\left(\frac{a+b}{2}\right)\right|^q} + \right.$$

$$\left.\sqrt[q]{|f'(b)|^q + 2\left|f'\left(\frac{a+b}{2}\right)\right|^q}\right]$$

Here by applying convexity on $\left|f'\left(\frac{a+b}{2}\right)\right|^q$, we obtained [6, Theorem 2.3].

Theorem 5. Let the assumptions of Theorem 2 be satisfied, then

$$\left|\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le (b-a)\times$$

$$\left[|f'(a)|\int_0^1\left|\frac{\mu}{\lambda+\mu}-t\right|h(t)\,dt + \right.$$

$$\left.|f'(b)|\int_0^1\left|t-\frac{\lambda}{\lambda+\mu}\right|h(t)\,dt+\right] \qquad (15)$$

Proof.

The proof is similar to proof of Theorem 2.

Remark 3. Setting *h*(*t*) = *t* and λ = μ ≠ 0. Theorem 5 coincides with [3, Theorem 2.2].

Theorem 6. Let the assumptions of Theorem 3 be satisfied, then

$$\left|\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le (b-a)\times$$

$$\left\{\frac{1}{2}-\frac{\lambda\mu}{\lambda+\mu}\right\}^{1/p}\left[|f'(a)|^q\int_0^1\left|\frac{\mu}{\lambda+\mu}-t\right|h(t)\,dt + \right.$$

$$\left.|f'(b)|^q\int_0^1\left|t-\frac{\lambda}{\lambda+\mu}\right|h(t)\,dt+\right]^{1/q} \qquad (16)$$

Proof.

By taking modulus on both sides of (6)

$$\left|\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le (b-a)\times$$

$$\int_0^1\left|\frac{\mu}{\lambda+\mu}-t\right|\;|f'(ta+(1-t)b)|\,dt \qquad (17)$$

By Hölder's inequality and *h*-convexity of $|f'|^q$:

$$\int_0^1\left|\frac{\mu}{\lambda+\mu}-t\right|\;|f'(ta+(1-t)b)|\,dt \le$$

$$\left\{\frac{1}{2}-\frac{\lambda\mu}{(\lambda+\mu)^2}\right\}^{1/p}\left[|f'(a)|^q\int_0^1\left|\frac{\mu}{\lambda+\mu}-t\right|h(t)\,dt+\right.$$

$$\left.|f'(b)|^q\int_0^1\left|\frac{\mu}{\lambda+\mu}-t\right|h(1-t)\,dt\right]^{1/q} \qquad (18)$$

To obtain (16), using (18) in (17)

Remark 4. Setting $h(t) = t$ and $\lambda = \mu \neq 0$. Theorem 5 coincides with [1, Theorem 48].

Theorem 7. Let the assumptions of Theorem 3 be satisfied, then

$$\left|\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)dx\right|\le(b-a)\times$$

$$\sqrt[p]{\frac{1}{p+1}\left\{\left(\frac{\lambda}{\lambda+\mu}\right)^{p+1}+\left(\frac{\mu}{\lambda+\mu}\right)^{p+1}\right\}}\times$$

$$\left[\left\{|f'(a)|^q+|f'(b)|^q\right\}\int_0^1 h(t)dt\right]^{1/q} \qquad (19)$$

Proof.
The proof is similar to proof of Theorem 4.

Remark 5. Setting $h(t) = t$ and $\lambda = \mu \neq 0$. Theorem 7 coincides with [3, Theorem 2.3].

Theorem 8. Let $f: I \subseteq \mathbb{R} \to \Re$ be a differentiable function on I^o such that $f' \in L[a,b]$ for $a,b \in I^o$ with $a<b$. If $|f'|^q \in SV(h,I)$

with $p=\dfrac{q}{q-1}$ for $q>1$ and λ, $\mu>0$, then

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{\mu a+\lambda b}{\lambda+\mu}\right)\right|\le$$

$$\frac{b-a}{\sqrt[p]{2h\left(\frac{1}{2}\right)}(\lambda+\mu)^2}\left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}}\times$$

$$\left[\lambda^2\left|f'\left(\frac{\mu a+\lambda\frac{a+b}{2}}{\lambda+\mu}\right)\right|+\mu^2\left|f'\left(\frac{\lambda b+\mu\frac{a+b}{2}}{\lambda+\mu}\right)\right|\right] \qquad (20)$$

Proof.

As $|f'|^q \in SV(h,I)$, therefore by Theorem 1

$$\int_0^1\left|f'\left(ta+\frac{(1-t)(\mu a+\lambda b)}{\lambda+\mu}\right)\right|^q dt\le$$

$$\frac{1}{2h(\frac{1}{2})}\left|f'\left(\frac{\mu a+\lambda\frac{a+b}{2}}{\lambda+\mu}\right)\right|^q \qquad (21)$$

Analogously

$$\int_0^1\left|f'\left(tb+\frac{(1-t)(\mu a+\lambda b)}{\lambda+\mu}\right)\right|^q dt\le$$

$$\frac{1}{2h(\frac{1}{2})}\left|f'\left(\frac{\lambda b+\mu\frac{a+b}{2}}{\lambda+\mu}\right)\right|^q \qquad (22)$$

A combination of (8), (13) – (14) and (21) – (22) yields (20)

Corollary 4. Under the assumptions of

Theorem 8 with $h(t) = t$ and $\lambda = \mu \neq 0$

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right|\le$$

$$\frac{b-a}{4\sqrt[p]{p+1}}\left[\left|f'\left(\frac{3a+b}{4}\right)\right|+\left|f'\left(\frac{a+3b}{4}\right)\right|\right]$$

Remark 6. For $h(t) = t^s$ where S $\in$ (0, 1], 1 and $\frac{1}{t}$ with $\lambda = \mu \neq 0$ relations (7), (9), (12), (15) – (16) and (19) – (20) provide the estimates of Hadamard differences for functions belong to $K_s^2, P(I)$ and $Q(I)$, respectively.

3. Applications to Some Special Means

a. The arithmetic mean

$$A = A(a,b) := \frac{a+b}{2}, \qquad a, b > 0$$

b. The geometric mean

$$G = G(a,b) := \sqrt{ab}, \qquad a, b > 0$$

c. The harmonic mean

$$H = H(a,b) := \frac{2ab}{a+b}, \qquad a, b > 0$$

d. The logarithmic mean

$$L = L(a,b) = \begin{cases} a & if\ a = b \\ \dfrac{b-a}{nb - \ln a} & if\ a \neq b \end{cases} \qquad \text{a, b > 0}$$

e. The identric mean

$$l = l(a,b) = \begin{cases} a & if\ a = b \\ \dfrac{1}{e}^{b-a}\sqrt{\dfrac{b^b}{a^a}} & if\ a \neq b \end{cases} \qquad \text{a, b > 0}$$

The n-logarithmic mean

$$L_n = L_n(a,b) = \begin{cases} a & if\ a = b \\ \sqrt[n]{\dfrac{b^{n+1} - a^{n+1}}{(n+1)(b-a)}} & if\ a \neq b \end{cases} \qquad \text{a, b > 0}$$

It is also known that L_n is monotonically increasing over, $n \in \Re$, denoting $L_o = L$ and $L_{-1} = L$. The following inequality is well known in the literature:

$$H \leq G \leq L \leq l \leq A$$

The following propositions hold:

Proposition 1. .Let $a, \quad b \in \mathbb{R} \qquad 0 < a < b$

then for all $q > 1$, we have

$$|L - G| \leq \frac{b-a}{4\sqrt[q]{3}} \times \frac{A\left(\sqrt[q]{a^q + G^q}, \sqrt[q]{b^q + G^q},\right)}{L}$$

Proof.

The proof follows by Theorem 3 by setting convex function $f(x) = e^x$ for

$\lambda = \mu \neq 0$

Proposition 2. .Let $a, \ b \in \mathbb{R} \qquad 0 < a < b$

Then for all $p > 1$ with $q = \frac{p}{p-1}$, we have

$$\ln\left(\frac{A}{l}\right) = \frac{b-a}{4}\sqrt[p]{\frac{2}{p+1}}A\left(\sqrt[q]{\frac{1}{a^q} + \frac{1}{A^q}}, \sqrt[q]{\frac{1}{b^q} + \frac{1}{A^q}},\right)$$

Proof

The proof follows by Theorem 4 by setting convex function $f(x) = -\ln x$ for

□ $\lambda = \mu \neq 0.$

Proposition 3. Let $a, b \in \mathbb{R} \quad 0 < a < b$

then for all $p > 1$ with $q = \frac{p}{p-1}$, we have

$$\left|H^{-1} - L^{-1}\right| \leq \frac{b-a}{4} \times \frac{1}{\sqrt[q]{H(a^{2q}, b^{2q})}}$$

Proof

The proof follows by Theorem 6 by setting convex function $f(x) = \frac{1}{x}$ for

$\lambda = \mu \neq 0.$

Proposition 4. Let $a, b \in \mathbb{R} \quad 0 < a < b$

And $n \in \mathbb{N}, \quad n > 2$. Then for all $p > 1$ with $q = \frac{p}{p-1}$, we have

$$\left|A(a^n, b^n) - L_n^n(a,b)\right| \leq$$

$$\frac{b-a}{2\sqrt[p]{n+1}}\sqrt[q]{nA(a^{n-1}, b^{n-1})}$$

Proof. The proof follows by Theorem 7 by setting convex function $f(x) = x^n$, $x \in \mathbb{R}$ for

$\lambda = \mu \neq 0.$

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